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LETTER TO THE EDITOR

An extremum property of the n-dimensional sphere

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Abstract. Using arguments by Krahn, it is shown that the n-dimensional sphere possesses the following minimum property: among all connected regions of the given volume it yields the smallest ground eigenvalue of the Laplace operator with Dirichlet's boundary condition. This result is important in establishing an adiabatic invariant for quantum mechanical systems.

The problem expounded in the Abstract was formulated (in its two-dimensional version) by Lord Rayleigh in the 19th century (1894), while the first proof of his hypothesis was given a few decades later (Faber 1923). Krahn (1925) has shown independently that a drum of given surface area must be of circular shape for the ground tone to be lowest. Similar extremal properties were studied by Poincaré (1902) for the capacity of conductors. Today, many different isoperimetric properties are known in mathematical physics (Polya and Szegö 1951, Vinogradov 1979).

The task of the present Letter is to find a connected domain D of unit volume in n-dimensional Euclidean space such that the lowest eigenvalue E, determined by the equation $\Delta\psi + E\psi = 0$ with $\psi = 0$ at ∂D, is minimal, i.e. it is larger for any other connected domain D of unit volume. Our variational functional may be written as

$$E = \int_D |\mathrm{grad}_q \psi|^2 \, \mathrm{d}^n q = E(D), \tag{1}$$

where constraints read

$$\psi|_{\partial D} = 0, \qquad \|\psi\|^2 := \int_D \psi^2 \, \mathrm{d}^n q = 1, \qquad \int_D \mathrm{d}^n q = 1.$$

Since we are studying the ground state solution, which is nodeless, we may assume that ψ is positive in the interior of D, and assumes all values between 0 (at the boundary) and its maximal value m (somewhere in the interior). We shall use the notation $W := |\mathrm{grad}_q \psi|$.

The problem is not merely of academic or formalistic interest, but is closely related to the existence of an adiabatic invariant for quantum mechanical systems (Robnik 1980). It turns out that the number of nodal cells into which the configuration space is decomposed by the nodal surfaces remains constant when the potential is continuously varied. This is implied by the fact that the volume of a nodal cell cannot be arbitrarily small as long as the energy is bounded: an upper limit for the energy implies a lower limit for the volume of a nodal cell.

Now we state that the minimal domain D is the n-dimensional sphere of unit volume. To prove this we translate arguments given by Krahn (1925).

Consider surfaces $\Omega(u)$ defined by $\psi(q)=u=$ constant. Let $R(u)$ be the region containing all points q from D such that $m\geqslant\psi(q)\geqslant u$. Hence $\Omega(u)$ is the boundary of $R(u)\subseteq D$. Moreover, if $u_1>u_2$, then $R(u_1)\subset R(u_2)$, which is a simple consequence of the fact that $\psi(q)$ is a single-valued function on D. By $S(u)$ we denote the surface area of $\Omega(u)$, i.e. $S(u):=\int_{\Omega(u)}\mathrm{d}S_u$, and by $V(u)$ the volume of the region $R(u)$, i.e. $V(u)=\int_{R(u)}\mathrm{d}^n q$. Now, in evalulating the integral (1) we integrate over the shells bounded by $\Omega(u)$ and $\Omega(u+\mathrm{d}u)$, i.e. the shells of constant ψ thickness, while their geometrical thickness at each point equals $\mathrm{d}u/W$. Hence

$$E=\int_0^m \mathrm{d}u\int_{\Omega(u)} W\,\mathrm{d}s_u, \tag{2}$$

while

$$V(u)=\int_u^m \mathrm{d}u\int_{\Omega(u)} W^{-1}\,\mathrm{d}S_u. \tag{3}$$

Consequently, $V(0)=1$, $V(m)=0$, $V'(u):=\mathrm{d}V/\mathrm{d}u=-\int_{\Omega(u)}W^{-1}\,\mathrm{d}S_u$.

Next consider the algebraic inequality $(x+a^2/x)^2\geqslant 4a^2$, with equality being valid iff $x=a$, and apply it to $x=1/\sqrt{W}$ and $a^2=-V'(u)/S(u)$. By integrating over the surface $\Omega(u)$ we obtain the inequality

$$\int_{\Omega(u)} W\,\mathrm{d}S_u\geqslant-\frac{S^2(u)}{V'(u)}. \tag{4}$$

Equality holds iff $W=-S(u)V'(u)$, so that $W=W(u)$ is independent of the position on $\Omega(u)$, i.e. the surfaces of constant ψ are at the same time the surfaces of constant W. Now we use the isoperimetric theorem: $S\geqslant kV^{1-1/n}$, i.e. the perimeter S of a connected domain of volume V is the smallest for the n-dimensional sphere (Vinogradov 1979). Here $k=n\sqrt{\pi}[\Gamma(n/2+1)]^{-1/n}$, where $\Gamma(x)$ is the gamma function. From this, (1) and (4) follows

$$E(D)\geqslant-k\int_0^m \mathrm{d}u\frac{V^{(2-2/n)}}{V'}, \tag{5}$$

where the equality holds only iff each $\Omega(u)$ is a sphere, in particular when $\Omega(u=0)=\partial D=n$-dimensional sphere.

Suppose (5) is violated for some $\phi(q)$ and D. Then we can construct another function $f(q)$ and D_0 such that all constraints of the functional (1) are obeyed and $V(u)$ is the same for each u, while $\Omega_f(u)$ for the function f is a sphere at each u. Then, in (4) the equality holds, and so it does in (5). But then $\phi(q)$ and D cannot exist, since this would contradict the isoperimetric theorem. QED

Finally we calculate explicitly the ground eigenvalue for the n-dimensional sphere of volume V. The Laplace operator reduces to a simple ordinary differential operator, and we have to solve the equation

$$r^{-(n-1)}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{(n-1)}\frac{\mathrm{d}\psi}{\mathrm{d}r}\right)+E\psi=0,$$

with $\psi(a)=0$, where a is the radius of the sphere. The solution reads

$$\psi=r^{1-n/2}J_{|1-n/2|}(r\sqrt{E}),$$

where $J_\nu(x)$ is the Bessel function of order ν. Thus, if its first zero is denoted by z_0, then

$E = z_0^2/a^2$. Since the volume of the sphere equals $v = a^n \pi^{n/2}/\Gamma(n/2+1)$, the final result reads

$$E = \pi z_0^2[\Gamma(n/2+1)]^{-2/n} V^{-2/n}. \tag{6}$$

For all other domains of volume V the ground eigenvalue is larger.

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